Normalization by evaluation for a language with algebraic effects (and their handlers)

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joint work with Sam Staton

Overview

- Fine-grain call-by-value (FGCBV) with algebraic effects
- Normalization by evaluation (NBE) for FGCBV with algebraic effects
- Problems in NBE for FGCBV with algebraic effects and their handlers

Fine-grain call-by-value

(FGCBV)

Value and producer terms

- Two typing judgments:
  - Values - $\Gamma \vdash_v V : \sigma$ Producers - $\Gamma \vdash_p M : \sigma$
- Value terms:

$$\frac{}{\Gamma, x : \sigma, \Gamma' \vdash_v x : \sigma} \qquad \frac{}{\Gamma \vdash_v \star : 1}$$

$$\frac{\Gamma \vdash_v V_1 : \sigma_1 \quad \Gamma \vdash_v V_2 : \sigma_2}{\Gamma \vdash_v \langle V_1, V_2 \rangle : \sigma_1 \times \sigma_2} \qquad \frac{\Gamma \vdash_v V : \sigma_1 \times \sigma_2}{\Gamma \vdash_v \pi_i(V) : \sigma_i}$$

$$\frac{\Gamma, x : \sigma \vdash_p N : \tau}{\Gamma \vdash_v \lambda x : \sigma . N : \sigma \rightharpoonup \tau}$$

- Producer terms:

$$\frac{\Gamma \vdash_v V : \sigma \rightharpoonup \tau \quad \Gamma \vdash_v W : \sigma}{\Gamma \vdash_p VW : \tau} \qquad \frac{\Gamma \vdash_v V : \sigma}{\Gamma \vdash_p \text{return } V : \sigma}$$

$$\frac{\Gamma \vdash_p M : \sigma \quad \Gamma, x : \sigma \vdash_p N : \tau}{\Gamma \vdash_p M \,\text{to}\, x . N : \tau}$$

Equational theory

- The usual and expected $\beta\eta$-equations from STLC, e.g.:

$$\frac{\Gamma, x : \sigma \vdash_p M : \tau \quad \Gamma \vdash_v V : \sigma}{\Gamma \vdash_p (\lambda x : \sigma . M) V \equiv M[V/x] : \tau}$$

$$\frac{\Gamma \vdash_v V : \sigma \rightharpoonup \tau}{\Gamma \vdash_v V \equiv \lambda x : \sigma . (V x) : \sigma \rightharpoonup \tau}$$

- And to/return-specific $\beta\eta$-equations, e.g.:

$$\frac{\Gamma \vdash_v V : \sigma \quad \Gamma, x : \sigma \vdash_p N : \tau}{\Gamma \vdash_p \mathsf{return}\, V\, \mathsf{to}\, x . N \equiv N[V/x] : \tau}$$

$$\frac{\Gamma \vdash_p M : \sigma \quad \Gamma, x : \sigma \vdash_v x : \sigma}{\Gamma \vdash_p M \equiv M\, \mathsf{to}\, x . \mathsf{return}\, x : \sigma}$$

Extending FGCBV with algebraic effects

- We are working with a variant of value and effect theories of Plotkin et. al.
- All base and arity types (e.g., Bool) define FGCBV types
- Every function symbol $\mathsf{f} : \beta_1, \dots, \beta_n \to \beta$ defines a value term constructor:

$$\frac{\Gamma \vdash_v V_1 : \beta_1 \quad \dots \quad \Gamma \vdash_v V_n : \beta_n}{\Gamma \vdash_v \mathsf{f}\,(V_1, \dots, V_n) : \beta}$$

- Every operation symbol $\mathsf{op} : \alpha \to \beta$ defines a producer term constructor:

$$\frac{\Gamma \vdash_v p : \beta \qquad \Gamma \vdash_p M_1 : \sigma \ \dots \ \Gamma \vdash_p M_n : \sigma}{\Gamma \vdash_p \mathsf{op}_p\, M_1 \dots M_n : \sigma}$$

(NB! we assume finite arities)

- Equations extend straightforwardly

Example: Global state with one location

- Base type Bool
- Two 0-ary function symbols true : Bool and false : Bool
- Operations:

$$\frac{\Gamma \vdash_p M : \sigma \qquad \Gamma \vdash_p N : \sigma}{\Gamma \vdash_p \mathsf{read}\ M\ N : \sigma} \qquad \frac{\Gamma \vdash_v V : \mathsf{Bool} \qquad \Gamma \vdash_p M : \sigma}{\Gamma \vdash_p \mathsf{write}_V\ M : \sigma}$$

- Equations:

$$\mathsf{write}_V\,(\mathsf{write}_{V'}\ M) \equiv \mathsf{write}_{V'}\ M$$

$$\mathsf{read}\,(\mathsf{read}\ M\ N)\ P \equiv \mathsf{read}\ M\ P$$

$$\mathsf{read}\ M\,(\mathsf{read}\ N\ P) \equiv \mathsf{read}\ M\ P$$

$$\mathsf{write}_{\mathsf{false}}\,(\mathsf{read}\ M\ N) \equiv \mathsf{write}_{\mathsf{false}}\ M$$

$$\mathsf{write}_{\mathsf{true}}\,(\mathsf{read}\ M\ N) \equiv \mathsf{write}_{\mathsf{true}}\ N$$

- Two algebraicity equations:

$$(\mathsf{read}\ M\ N)\ \mathsf{to}\ x.P \equiv \mathsf{read}\,(M\ \mathsf{to}\ x.P)\,(N\ \mathsf{to}\ x.P)$$

$$(\mathsf{write}_V\ M)\ \mathsf{to}\ x.P \equiv \mathsf{write}_V\,(M\ \mathsf{to}\ x.P)$$

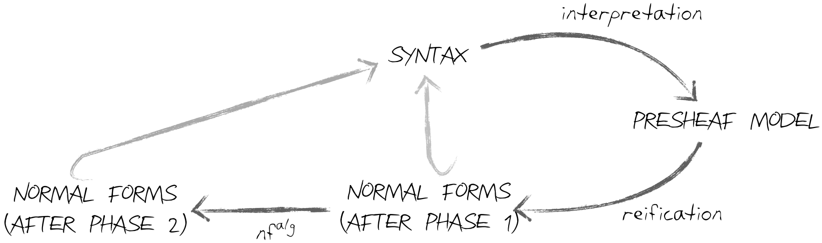
NBE for FGCBV

with algebraic effects

Two phases of normalization

- We have divided the normalization problem into two phases:

1. Phase 1: Normalize the underlying FGCBV language
   - Normal forms are identified up to the equations in effect theory
2. Phase 2: If necessary/needed/possible normalize the algebraic effects



Phase 1: Normal forms

- The normal and atomic (neutral) forms:

$$\frac{}{\Gamma, x : \sigma, \Gamma' \vdash_{av} x : \sigma}$$

$$\frac{\Gamma \vdash_{av} V : \mathrm{Bool}}{\Gamma \vdash_{nv} V : \mathrm{Bool}}$$

$$\frac{}{\Gamma \vdash_{nv} \star : 1}$$

$$\frac{\Gamma \vdash_{nv} V_1 : \sigma_1 \quad \Gamma \vdash_{nv} V_2 : \sigma_2}{\Gamma \vdash_{nv} \langle V_1, V_2 \rangle : \sigma_1 \times \sigma_2}$$

$$\frac{}{\Gamma \vdash_{nv} \mathrm{true}/\mathrm{false} : \mathrm{Bool}}$$

$$\frac{\Gamma \vdash_{av} V : \sigma_1 \times \sigma_2}{\Gamma \vdash_{av} \pi_i(V) : \sigma_i}$$

$$\frac{\Gamma, x : \sigma \vdash_{np} N : \tau}{\Gamma \vdash_{nv} \lambda x : \sigma . N : \sigma \rightharpoonup \tau}$$

$$\frac{\Gamma \vdash_{nv} V : \sigma}{\Gamma \vdash_{np} \mathrm{return}\ V : \sigma}$$

$$\frac{\Gamma \vdash_{ap} M : \sigma \quad \Gamma, x : \sigma \vdash_{np} N : \tau}{\Gamma \vdash_{np} M\ \mathrm{to}\ x . N : \tau}$$

$$\frac{\Gamma \vdash_{av} V : \sigma \rightharpoonup \tau \quad \Gamma \vdash_{nv} W : \sigma}{\Gamma \vdash_{ap} VW : \tau}$$

$$\frac{\Gamma \vdash_{np} M : \sigma \qquad \Gamma \vdash_{np} N : \sigma}{\Gamma \vdash_{np} \mathrm{read}\ M\ N : \sigma}$$

$$\frac{\Gamma \vdash_{nv} V : \mathrm{Bool} \qquad \Gamma \vdash_{np} M : \sigma}{\Gamma \vdash_{np} \mathrm{write}_V\ M : \sigma}$$

Phase 1: Equational theory of normal forms

- Propositional equality between atomic values
- Equations between normal values, atomic producers, normal producers
  - reflexivity, symmetry, transitivity
  - congruence of term constructors
- In addition, equations for normal producers include the effect theory

$$\mathsf{write}_V\,(\mathsf{write}_{V'}\,M) \equiv \mathsf{write}_{V'}\,M$$

$$\mathsf{read}\,(\mathsf{read}\,M\,N)\,P \equiv \mathsf{read}\,M\,P$$

$$\mathsf{read}\,M\,(\mathsf{read}\,N\,P) \equiv \mathsf{read}\,M\,P$$

$$\mathsf{write}_{\mathsf{false}}\,(\mathsf{read}\,M\,N) \equiv \mathsf{write}_{\mathsf{false}}\,M$$

$$\mathsf{write}_{\mathsf{true}}\,(\mathsf{read}\,M\,N) \equiv \mathsf{write}_{\mathsf{true}}\,N$$

Phase 1: Semantics

- We use PERs over presheaves $\mathrm{Set}^{\mathrm{Ctx}^{op}}$

  (see also Cubric, Dybjer, Scott (1998))
- Types are interpreted as:
  - $[\![\mathrm{Bool}]\!]\,\Gamma = \Gamma \vdash_{nv} \mathrm{Bool}$

    $[\![1]\!]\,\Gamma = \mathrm{Unit}$

    $[\![\sigma_1 \times \sigma_2]\!]\,\Gamma = [\![\sigma_1]\!]\,\Gamma \times [\![\sigma_2]\!]\,\Gamma$

    $[\![\sigma \rightharpoonup \tau]\!]\,\Gamma = ([\![\sigma]\!] \Rightarrow T\,[\![\tau]\!])\,\Gamma$
- Where the necessary monad T turned out to be just the free term algebra monad:

  data T (X : Ctx $\rightarrow$ *Set*) : *Ctx* $\rightarrow$ *Set where*

  T-return : $\mathrm{X}\,\Gamma \rightarrow T\,X\,\Gamma$

  T-to : $\Gamma \vdash_{ap} \sigma \rightarrow T\,X\,(\Gamma :: \sigma) \rightarrow T\,X\,\Gamma$

  T-read : $\mathrm{T}\,\mathrm{X}\,\Gamma \rightarrow T\,X\,\Gamma \rightarrow T\,X\,\Gamma$

  T-write : $\Gamma \vdash_{nv} \mathit{Bool} \rightarrow T\,X\,\Gamma \rightarrow T\,X\,\Gamma$

Phase 1: Partial equivalence relations

- The PER for for semantic values is generated by (recursive definition):
  - $d \approx^{\Gamma;\,\mathrm{Bool}} d' \iff \Gamma \vdash_{nv} d \equiv d'$
  - $d \approx^{\Gamma;\,One} d' \iff d \cong d'$
  - $d \approx^{\Gamma;\,\sigma_1 \wedge \sigma_2} d' \iff$
    $(fst\ d) \approx^{\Gamma;\,\sigma_1} (fst\ d')\ \&\&\ (snd\ d) \approx^{\Gamma;\,\sigma_1} (snd\ d')$
  - $d \approx^{\Gamma;\,\sigma \rightharpoonup \tau} d' \iff (\forall f \in (\mathrm{Ren}\,\Gamma\,\Gamma'), d'', d''' \in ([\![\sigma]\!]\,\Gamma')\,.$
    d" $\approx^{\Gamma';\,\sigma} d''' \implies (d\ f\ d'') \approx_T^{\Gamma';\,\tau} (d'\ f\ d'''))$
- and for the monad by (inductive definition):
  - symmetry, transitivity
  - congruence for T-return, T-to, T-read, T-write
  - and the equations from the effect theory

Phase 1: Interpretation

- $[\![\_]\!]_v : \{\Gamma : Ctx\}\{\sigma : Ty\} \to \Gamma \vdash_v \sigma \to \{\Gamma' : Ctx\} \to Env\,\Gamma\,\Gamma' \to [\![\sigma]\!]\,\Gamma'$

  $[\![\_]\!]_p : \{\Gamma : Ctx\}\{\sigma : Ty\} \to \Gamma \vdash_p \sigma \to \{\Gamma' : Ctx\} \to Env\,\Gamma\,\Gamma' \to T\,[\![\sigma]\!]\,\Gamma'$

  $[\![x]\!]_v\, e = e\, x$

  ...

  $[\![\lambda x : \sigma\, .\, N]\!]_v\, e = \lambda d\, .\, [\![N]\!]_p\, (e[d/x])$

  ...

  $[\![\mathrm{return}\ V]\!]_p\, e = \eta\, ([\![V]\!]_v\, e)$

  ...

  $[\![M\,\mathrm{to}\,x.N]\!]_p\, e = (\lambda e, d\, .\, [\![N]\!]_p\, (e[d/x]))^*\, (\mathrm{str}\, (e, [\![M]\!]_p\, e))$

  ...

  $[\![\mathrm{read}\ M\ N]\!]_p\, e = T{-}read\, ([\![M]\!]_p\, e)\, ([\![N]\!]_p\, e)$

  ...

  $[\![\mathrm{write}_V\, M]\!]_p\, e = T{-}write\, ([\![V]\!]_v\, e)\, ([\![M]\!]_p\, e)$

Phase 1: Soundness

- **Theorem:**
  If $\Gamma \vdash_v V \equiv W : \sigma$ then $[\![V]\!]_v\, e \approx^{\Gamma;\sigma} [\![W]\!]_v\, e'$

  and

  if $\Gamma \vdash_p M \equiv N : \sigma$ then $[\![M]\!]_p\, e \approx_T^{\Gamma;\sigma} [\![N]\!]_p\, e'$

- Where e and e' are environments related by a PER induced by $\approx^{\Gamma;\sigma}$ at every variable

- Proof uses Kripke logical relations between terms and their denotations to capture that equivalent terms have same (here equivalent) denotations
  - $\sim_v^\sigma \in (\Gamma \vdash_v \sigma) \times ([\![\sigma]\!]\,\Gamma)$ *(by recursion on type structure)*
  - $\sim_p^\sigma \in (\Gamma \vdash_p \sigma) \times (T\,[\![\sigma]\!]\,\Gamma)$ *(by recursion on monad structure)*

Phase 1: Reification and reflection

- We define four (2 x reify + 2 x reflect) mutually recursive functions:
  - $\text{reify}^v_\sigma : \{\Gamma : Ctx\} \to [\![\sigma]\!]\,\Gamma \to \Gamma \vdash_{nv} \sigma$

    $\text{reify}^v_{\text{One}} \star = \star$

    $\text{reify}^v_{\text{Bool}}\, V = V$

    $\text{reify}^v_{\sigma_1 \wedge \sigma_2}\, d = \langle \mathit{reify}^v_{\sigma_1}\, (\mathit{fst}\ d)\,,\ \mathit{reify}^v_{\sigma_2}\, (\mathit{snd}\ d) \rangle$

    $\text{reify}^v_{\sigma \rightharpoonup \tau} d = \lambda x : \sigma\,.\,(\mathit{reify}^p_\tau\,(d\ \mathit{weaken}\,(\mathit{reflect}^v_\sigma\ x)))$

    $\text{reify}^p_\sigma : \{\Gamma : Ctx\} \to T\,[\![\sigma]\!]\,\Gamma \to \Gamma \vdash_{np} \sigma$

    $\text{reify}^p_\sigma\,(T\!-\!\mathit{return}\,d) = \text{return}\,(\mathit{reify}^v_\sigma\ d)$

    $\text{reify}^p_\sigma\,(T\!-\!\mathit{to}\ M\ d) = M\,\text{to}\,x.(\mathit{reify}^p_\sigma\ d)$

    $\text{reify}^p_\sigma\,(T\!-\!\mathit{read}\ d\ d') = \text{read}\,(\mathit{reify}^p_\sigma\ d)\,(\mathit{reify}^p_\sigma\ d')$

    $\text{reify}^p_\sigma\,(T\!-\!\mathit{write}\ V\ d) = \text{write}_V\,(\mathit{reify}^p_\sigma\ t)$

Phase 1: Reification and reflection ctd.

- We define four (2 x reify + 2 x reflect) mutually recursive functions:
  - $\text{reflect}^v_\sigma : \{\Gamma : Ctx\} \to \Gamma \vdash_{av} \sigma \to [\![\sigma]\!]\,\Gamma$

    $\text{reflect}^v_{\text{One}}\,\star = \star$

    $\text{reflect}^v_{\text{Bool}}\,V = V$

    $\text{reflect}^v_{\sigma_1 \wedge \sigma_2}\,V = (\mathit{reflect}^v_{\sigma_1}\,(\pi_1\,V)\,,\ \mathit{reflect}^v_{\sigma_2}\,(\pi_2\,V))$

    $\text{reflect}^v_{\sigma \rightharpoonup \tau}\,V = \lambda d.\,\mathit{reflect}^p_\tau\,(V\,(\mathit{reify}^v_\sigma\,d))$

    $\text{reflect}^p_\sigma : \{\Gamma : Ctx\} \to \Gamma \vdash_{ap} \sigma \to T\,[\![\sigma]\!]\,\Gamma$

    $\text{reflect}^p_\sigma\,M = T\!-\!to\,M\,(T\!-\!return\,(\mathit{reflect}^v_\sigma\,x))$

- The normalization functions are then given by
  - $\text{nf}^v_\sigma V = \mathit{reify}^v_\sigma\,([\![V]\!]_v\,(\lambda x : \tau\,.\,\mathit{reflect}^v_\tau\,x))$

    $\text{nf}^p_\sigma M = \mathit{reify}^p_\sigma\,([\![M]\!]_p\,(\lambda x : \tau\,.\,\mathit{reflect}^v_\tau\,x))$

Phase 1: Normalization results

- **Theorem 1:** If $\Gamma \vdash_{nv} V : \sigma$ then $nf^{v}_{\sigma}\,(embed\ V) \cong V$

- **Theorem 2:**
  If $\Gamma \vdash_{v} V : \sigma$ then $\Gamma \vdash_{v} V \equiv embed\,(nf^{v}_{\sigma}\ V) : \sigma$

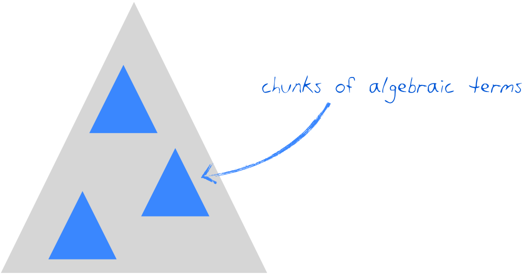
- **Theorem 3:**
  If $\Gamma \vdash_{v} V \equiv W : \sigma$ then $\Gamma \vdash_{nv} (nf^{v}_{\sigma}\ V) \equiv (nf^{v}_{\sigma}\ W) : \sigma$

- Symmetric results for producers.

Phase 2

- If we want and can normalize the effect theory:
  - We assume a given normalization algorithm nf^{alg} for the effect theory
  - Example: the theory of global state with one location has a straightforward recursive normalizer

- After normalizing the underlying language it is very convenient to apply nf^{alg} because:



Phase 2: Chunks of algebraic terms

- Example:

$$\mathsf{write}_V\,(\mathsf{return}\ x)\ \mathsf{to}\ y.(\mathsf{write}_{V'}\,(yz\ \mathsf{to}\ w.\\ (\mathsf{write}_{\mathsf{true}}\,(\mathsf{read}\ \,(\mathsf{return\ true})\,(\mathsf{return\ false})))))$$

$$\downarrow$$

$$\mathsf{write}_V\,(\mathsf{write}_{V'}\,(xz\ \mathsf{to}\ w.(\mathsf{write}_{\mathsf{true}}\,(\mathsf{read}\ \,(\mathsf{return\ true})\,(\mathsf{return\ false})))))$$

$$\downarrow$$

$$\mathsf{write}_{V'}\,(xz\ \mathsf{to}\ w.(\mathsf{write}_{\mathsf{true}}\,(\mathsf{return\ false})))$$

Phase 2: Chunks of algebraic terms ctd.

- We can give a straightforward typing system for algebraic and non-algebraic normal producers:

$$\frac{\Gamma \vdash_{nv} V : \sigma}{\Gamma \vdash_{nonalg} \text{return } V : \sigma} \qquad \frac{\Gamma \vdash_{ap} M : \tau \quad \Gamma, x : \tau \vdash_{np} N : \sigma}{\Gamma \vdash_{nonalg} M \,\text{to}\, x.N : \sigma}$$

$$\frac{\Gamma \vdash_{nonalg} M : \sigma}{\Gamma \vdash_{alg} \text{nonalg}\, M : \sigma}$$

$$\frac{\Gamma \vdash_{nv} V : \text{Bool} \quad \Gamma \vdash_{alg} M : \sigma}{\Gamma \vdash_{alg} \text{write}_V\, M : \sigma} \qquad \frac{\Gamma \vdash_{alg} M : \sigma \quad \Gamma \vdash_{alg} N : \sigma}{\Gamma \vdash_{alg} \text{read}\, M\, N : \sigma}$$

- Notice: We can think of the non-algebraic language terms as effect variables

Phase 2: Chunks of algebraic terms ctd.

- First, we lift nf^{alg} from the effect theory to algebraic language terms
  - $nf^{alg}_{\sigma} : \{\Gamma : Ctx\} \to \Gamma \vdash_{alg} \sigma \to \Gamma \vdash_{alg} \sigma$
- Next, we can characterize phase 2 by mutually defined functions:
  - $np\_to\_alg_{\sigma} : \{\Gamma : Ctx\} \to \Gamma \vdash_{np} \sigma \to \Gamma \vdash_{alg} \sigma$
  - $alg\_to\_np_{\sigma} : \{\Gamma : Ctx\} \to \Gamma \vdash_{alg} \sigma \to \Gamma \vdash_{np} \sigma$
  - $nf^{v\prime}_{\sigma} : \{\Gamma : Ctx\} \to \Gamma \vdash_{nv} \sigma \to \Gamma \vdash_{nv} \sigma$
  - $nf^{p\prime}_{\sigma} : \{\Gamma : Ctx\} \to \Gamma \vdash_{np} \sigma \to \Gamma \vdash_{np} \sigma$

Conclusions

- Normalization by evaluation for FGCBV extended with algebraic effects + formalization in Agda
- Adding handlers to NBE and FGCBV raised more interesting questions:
  - have to make compromises in the equational theory to be decidable
  - to define normal forms we need a “depth-measure” on continuation variables
- NBE for FGCBV with algebraic effects and handlers is work in progress

Additional material:

Notes about handlers in FGCBV and NBE

"To computations" or "to values"

- Assume empty value theory and only one operation symbol $op : 1 \to 1$
- To computations:

$$\frac{\Gamma \vdash_p M : \sigma \quad \Gamma, x : 1 \rightharpoonup \tau \vdash_p H_{op} : \tau \quad \Gamma, x : \sigma \vdash_p H_{ret} : \tau}{\Gamma \vdash_p \mathsf{handle}_p\, M\, \mathsf{with}\, \{\mathsf{op}(x) \Rightarrow H_{op} \mid \mathsf{return}\,(x) \Rightarrow H_{ret}\} : \tau}$$

- To values:

$$\frac{\Gamma \vdash_p M : \sigma \quad \Gamma, x : \tau \vdash_v H_{op} : \tau \quad \Gamma, x : \sigma \vdash_v H_{ret} : \tau}{\Gamma \vdash_v \mathsf{handle}_v\, M\, \mathsf{with}\, \{\mathsf{op}(x) \Rightarrow H_{op} \mid \mathsf{return}\,(x) \Rightarrow H_{ret}\} : \tau}$$

- The latter subsumes the former by

$$\mathsf{handle}_p\, M\, \mathsf{with}\, \{\mathsf{op}(x) \Rightarrow H_{op} \mid \mathsf{return}\,(x) \Rightarrow H_{ret}\} =$$
$$(\mathsf{handle}_v\, M\, \mathsf{with}\, \{\mathsf{op}(x) \Rightarrow \lambda \star . H_{op} \mid \mathsf{return}\,(x) \Rightarrow \lambda \star . H_{ret}\})\, \star$$

Redundancy of monadic let/to

- Obviously, the to/let binding becomes redundant:
- $M\,\mathsf{to}\,x.N =$
  $$\mathsf{handle}_p\,M\,\mathsf{with}\,\{\mathsf{op}\,(x) \Rightarrow \mathsf{op}(x\,\star)\,|\,\mathsf{return}\,(x) \Rightarrow N\}$$
- But what about the equational theory?
  - The defined to/let binding has to be "compatible" with the previous explicit one.

Equations for handlers ("to computations" case)

- Every handler induces a mediating morphism $T\,\sigma \to T\,\tau$ which is given by
  - $f : 1 \rightharpoonup \sigma \vdash_p \text{handle}_p\,(f\,\star)\,\text{with}\,H : \tau$

  (abbreviation $T\,\sigma = 1 \rightharpoonup \sigma$)

  which has to exhibit $T\,\sigma$ as the free algebra on σ, i.e.,

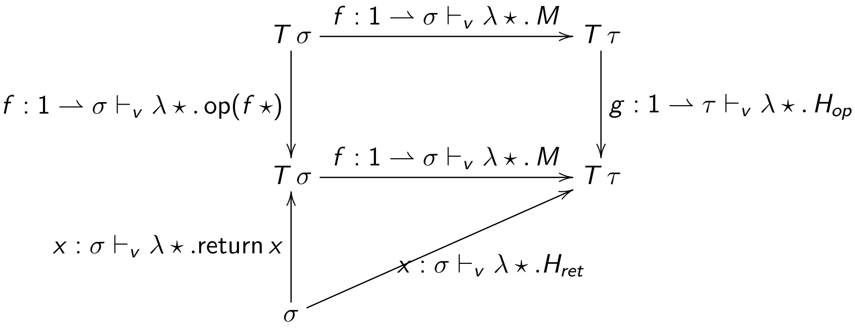
- It has to be a homomorphism of algebras:
  - $\Gamma \vdash_p \text{handle}_p\,(\text{return}\,V)\,\text{with}\,H \equiv H_{ret}[V/x] : \tau$
  - $\Gamma \vdash_p \text{handle}_p\,(\text{op}(M))\,\text{with}\,H \equiv$ $H_{op}[\lambda\star.\,(\text{handle}_p\,M\,\text{with}\,H)/x] : \tau$

Equations for handlers ("to computations" case)

- It also has to be unique amongst such morphisms, i.e.:
- $f : 1 \rightharpoonup \sigma \vdash_p M \equiv \mathsf{handle}_p\,(f\,\star)\,\mathsf{with}\,H : \tau$

  must hold if the following diagram commutes

  (abbreviation $T\,\sigma = 1 \rightharpoonup \sigma$)



System T with unique iteration

→ undecidability result by Okada&Scott 1991

Equations for handlers ("to computations" case)

- Due to Okada&Scott's result, we currently confine ourselves to the following η and associativity equations:
- $\Gamma \vdash_p \mathsf{handle}_p\, M\, \mathsf{with}$
  $$\{\mathsf{op}\,(x) \Rightarrow \mathsf{op}\,(x\,\star) \mid \mathsf{return}\,(x) \Rightarrow \mathsf{return}\,x\} \equiv M : \sigma$$
- $\Gamma \vdash_p$
  $\mathsf{handle}_p\,(\mathsf{handle}_p\, M\, \mathsf{with}\, \{\mathsf{op}\,(x) \Rightarrow \mathsf{op}\,(x\,\star) \mid \mathsf{return}\,(x) \Rightarrow N\})\, \mathsf{with}\, \{\mathsf{op}\,(y) \Rightarrow \mathsf{op}\,(y\,\star) \mid \mathsf{return}\,(y) \Rightarrow P\}$
  $\equiv$
  $\mathsf{handle}_p\, M\, \mathsf{with}\, \{\mathsf{op}\,(x) \Rightarrow \mathsf{op}\,(x\,\star) \mid \mathsf{return}\,(x) \Rightarrow (\mathsf{handle}_p\, N\, \mathsf{with}\, \{\mathsf{op}\,(y) \Rightarrow \mathsf{op}\,(y\,\star) \mid \mathsf{return}\,(y) \Rightarrow P\})\}$
  $: \sigma$

Normal forms for handlers

- Similarly to STLC with coproducts, we now have two separate judgments for normal producers:

$$\frac{\Gamma \vdash_{ap} M : \sigma}{\Gamma \vdash_{np'} M : \sigma} \qquad \frac{\Gamma \vdash_{np'} M : \sigma \quad \Gamma \vdash_{nh} H : \sigma,\tau}{\Gamma \vdash_{np'} \mathsf{handle}_p\, M \,\mathsf{with}\, H : \tau}$$

$$\frac{\Gamma \vdash_{nv} V : \sigma}{\Gamma \vdash_{np} \mathsf{return}\, V : \sigma} \qquad \frac{\Gamma \vdash_{np} M : \sigma}{\Gamma \vdash_{np} \mathsf{op}\, M : \sigma}$$

$$\frac{\Gamma \vdash_{np'} M : \sigma \quad \Gamma \vdash_{nh} H : \sigma,\tau}{\Gamma \vdash_{np} \mathsf{handle}_p\, M \,\mathsf{with}\, H : \sigma}$$

$$\frac{\Gamma, x : 1 \rightharpoonup \tau \vdash_{np} H_{op} : \tau \quad \Gamma, x : \sigma \vdash_{np} H_{ret} : \tau}{\Gamma \vdash_{nh} \{\mathsf{op}\,(x) \Rightarrow H_{op} \,|\, \mathsf{return}\,(x) \Rightarrow H_{ret}\} : \sigma;\tau}$$

Extending NBE with handlers

- Define semantic algebra structure corresponding to H_{op} and H_{ret}
  - $Han_{\sigma,\tau}\,\Gamma = (\llbracket 1 \rightharpoonup \tau \rrbracket \Rightarrow T\,\llbracket \tau \rrbracket)\,\Gamma \times (\llbracket \sigma \rrbracket \Rightarrow T\,\llbracket \tau \rrbracket)\,\Gamma$
- Extend the free term algebra monad with straightforward handling construct
  - Also need additional term algebra monad for $\vdash_{nh'}$
- Define the semantic mediating morphism
  - $med\_mor_{\sigma,\tau} : \{\Gamma : Ctx\} \rightarrow Han_{\sigma,\tau}\,\Gamma \rightarrow T\,\llbracket \sigma \rrbracket\,\Gamma \rightarrow T\,\llbracket \tau \rrbracket\,\Gamma$
- Extend the interpretation
  - $\llbracket \text{handle}\, M \,\text{with}\, H \rrbracket_p\, e = med\_mor\,(\llbracket H \rrbracket_h\, e)\,(\llbracket M \rrbracket_p\, e)$
  - $\llbracket H \rrbracket_h\, e = (\lambda d.\,(\llbracket H_{op} \rrbracket_p\,(e[d/x]))\,,\ \lambda d.\,(\llbracket H_{ret} \rrbracket_p\,(e[d/x])))$
- Extend reification and reflection functions
  $\rightarrow$ but there is a significant problem with normalizing $\vdash_{ap}$

Problem with computing normal forms in NBE

- Consider an atomic producer $\Gamma \vdash_{ap} M : \sigma$
- When we want it's normal form to be consistent with what we had in FGCBV without handlers we have:
  - $nf^{p}_{\sigma}\ M =$
    $\text{handle}\ M\ \text{with}\ \{\text{op}\,(x) \Rightarrow \text{op}\,(x\,\star) \mid \text{return}\,(x) \Rightarrow \text{return}\ x\}$
- But for $\text{op}\,(x\,\star)$ to be normal, $(x\,\star)$ has to be normal.
- But $(x\,\star)$ is again a atomic producer which needs normalizing.
- Anybody notices a problem with computing such η-long normal forms?